Exercise 88

Air is being pumped into a spherical weather balloon. At any time t, the volume of the balloon is $V(t)$ and its radius is $r(t)$.

(a) What do the derivatives dV/dr and dV/dt represent?

(b) Express dV/dt in terms of dr/dt.

Solution

dV/dt is the rate that volume changes as time increases, whereas dV/dr is the rate that volume changes as the radius increases. The volume of a sphere is known to be

$$V = \frac{4}{3}\pi r^3.$$

If the radius changes as a function of time, that is, $r = r(t)$, then

$$V = \frac{4}{3}\pi[r(t)]^3.$$

Differentiate the volume with respect to t.

$$\begin{aligned}
\frac{dV}{dt} &= \frac{d}{dt}\left\{\frac{4}{3}\pi[r(t)]^3\right\} \\
&= \frac{4}{3}\pi\frac{d}{dt}\left\{[r(t)]^3\right\} \\
&= \frac{4}{3}\pi\left\{3[r(t)]^2 \cdot \frac{d}{dt}[r(t)]\right\} \\
&= \frac{4}{3}\pi\left\{3[r(t)]^2 \cdot \frac{dr}{dt}\right\} \\
&= 4\pi[r(t)]^2\frac{dr}{dt}
\end{aligned}$$